Transmathematics

Dr James Anderson
安德生

Topics

- Why computer scientists like totality:
- Transarithmetic - no error states
- Transphysics - works at singularities
- Transcomputing - faster, safer, more accurate
- Summary

Totallity

- Totality - every function is a total function
- It is not necessary to waste states - so hardware and software can be more efficient
- Every syntactically correct program is semantically correct - except for physical intervention, programs cannot crash

Totallity

- Geometrical construction of the transreal numbers (1997)
- Axiomatisation and machine proof of the consistency of transreal arithmetic (2007)
- Human proof of the consistency of transreal and transcomplex arithmetic (2014, 2016)
- Current research on the foundations and applications of transmathematics

Arithmetica Universalis

Sir Isaac Newton

Arithmetica Universalis

- Newton used the arithmetica universalis, which predates real arithmetic
- Arithmetica universalis does not outlaw division by zero
- Arithmetica universalis allows a number to have any factors. For example, $0 = 0 \times 1 \times 2 \times \dots$ has factors $0,\ 1,\ 2,\ \dots$ on the right hand side
- Modern integer factors are required to be smaller than the number they factorise

Arithmetica Universalis

- Newton gave a faulty proof that division by zero is inconsistent

- Newton defined $x = y \iff x - y = 0$

- This blocks transreal $\pm\infty, \Phi$ because $(-\infty) - (-\infty) = \infty - \infty = \Phi - \Phi = \Phi$

- If we read Newton's equal (aequo) as a modern equality and restrict division by zero to transreal division by zero then the arithmetica universalis is transreal arithmetic and Newton's mechanics (physics) work at singularities!

Transreal Numbers

Transreal numbers, t, are proper fractions of real numbers, with a non-negative denominator, d, and a numerator, n, that is one of -1, 0, 1 when d = 0

$$t = \frac{n}{d}$$

With k a positive constant:

$$-\infty = \frac{-k}{0} = \frac{-1}{0} \qquad \Phi = \frac{0}{0} \qquad \infty = \frac{k}{0} = \frac{1}{0}$$

Improper Fractions

Improper fractions have a negative denominator (-k) which must be made positive *before* any arithmetical operator is applied

$$\frac{n}{-k} = \frac{-n}{-(-k)} = \frac{-1 \times n}{-1 \times (-k)} = \frac{-n}{k}$$

Multiplication

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

Division

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

Addition of Two Infinities

$$\infty + \infty = \frac{1}{0} + \frac{1}{0} = \frac{1+1}{0} = \frac{2}{0} = \frac{1 \times 2}{0 \times 2} = \frac{1}{0} = \infty$$

$$\infty + (-\infty) = \frac{1}{0} + \frac{-1}{0} = \frac{1-1}{0} = \frac{0}{0} = \Phi$$

$$-\infty + \infty = \frac{-1}{0} + \frac{1}{0} = \frac{-1+1}{0} = \frac{0}{0} = \Phi$$

$$-\infty + (-\infty) = \frac{-1}{0} + \frac{-1}{0} = \frac{(-1)+(-1)}{0} = \frac{-2}{0} = \frac{(-1) \times 2}{0 \times 2} = \frac{-1}{0} = -\infty$$

General Addition

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Subtraction

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d}$$

Associativity

$$a + (b + c) = (a + b) + c$$

$$a \times (b \times c) = (a \times b) \times c$$

Commutativity

$$a + b = b + a$$

$$a \times b = b \times a$$

Partial Distributivity

$$a(b+c) = ab + ac$$

When $a \neq \pm\infty$ or

$$bc > 0$$ or

$$(b+c)/0 = \Phi$$

Real Arithmetic versus Transreal Arithmetic

- Real arithmetic checks for division by zero and, if found, it fails
- Transreal arithmetic checks for division by zero and always succeeds

Newtonian Physics

- Newton used the arithmetica universalis
- The arithmetica universalis can be restricted to real arithmetic by outlawing division by zero
- The arithmetica universalis can be extended to transreal arithmetic by using a modern equality and restricting division by zero to transreal division by zero

Newtonian Physics

- If we read Newton's 18th Century physics as applying to real numbers then they fail at singularities
- If we read Newton's 18th Century physics as applying to transreal numbers then they operate at singularities

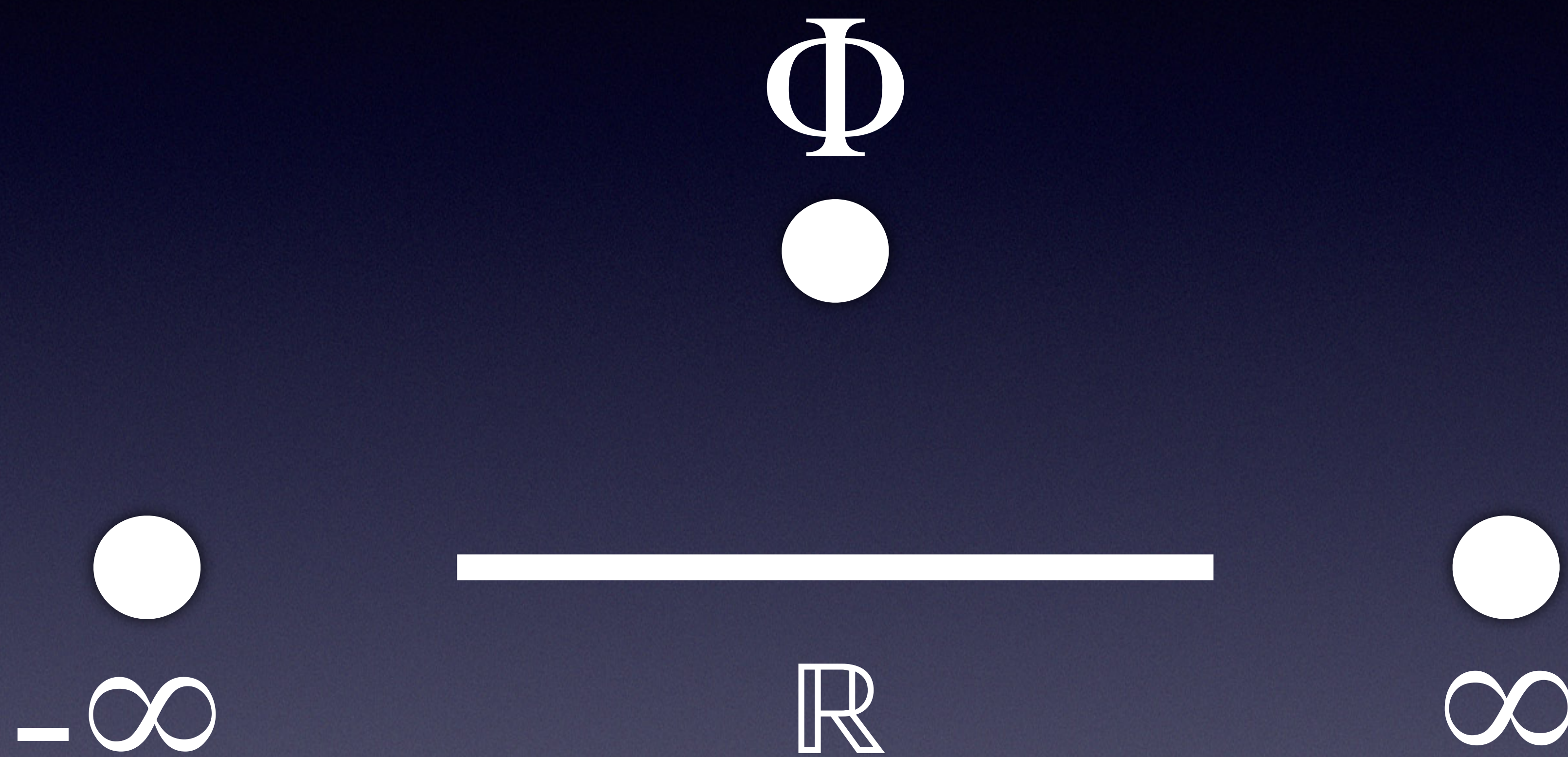
Transreal Analysis

- Transreal analysis replaces the symbols $-\infty, \infty$ of real analysis with the transreal numbers $-\infty = \dfrac{-1}{0}$ and $\infty = \dfrac{1}{0}$
- Every real result of real analysis arises as the same real result of transreal analysis
- Transreal analysis has some results that cannot be obtained by real analysis

Trans-Newtonian Physics

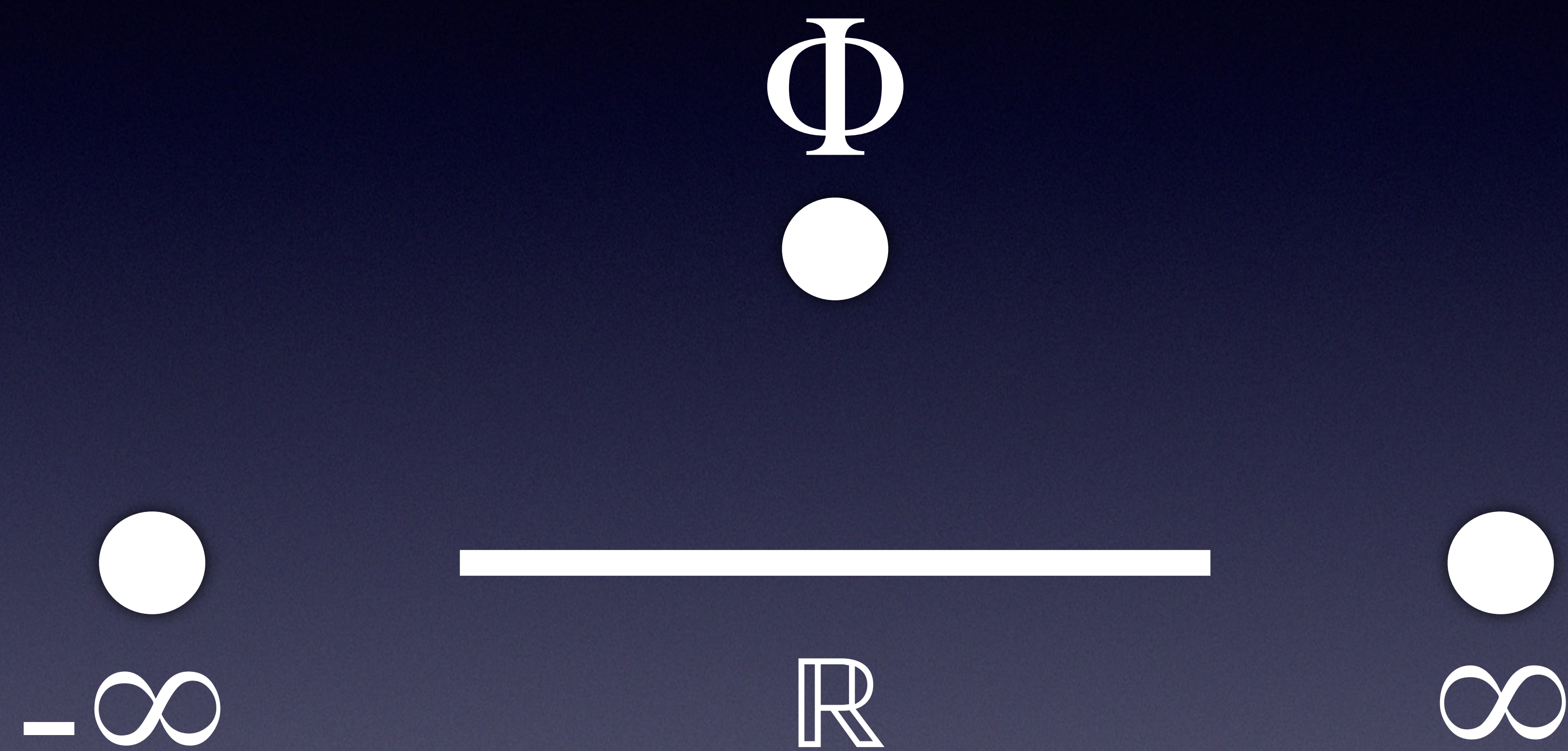
- Newton's laws of motion and gravity can be restated with transreal arithmetic and transreal analysis
- Hence Trans-Newtonian physics operates at singularities

Transreal-Number Line



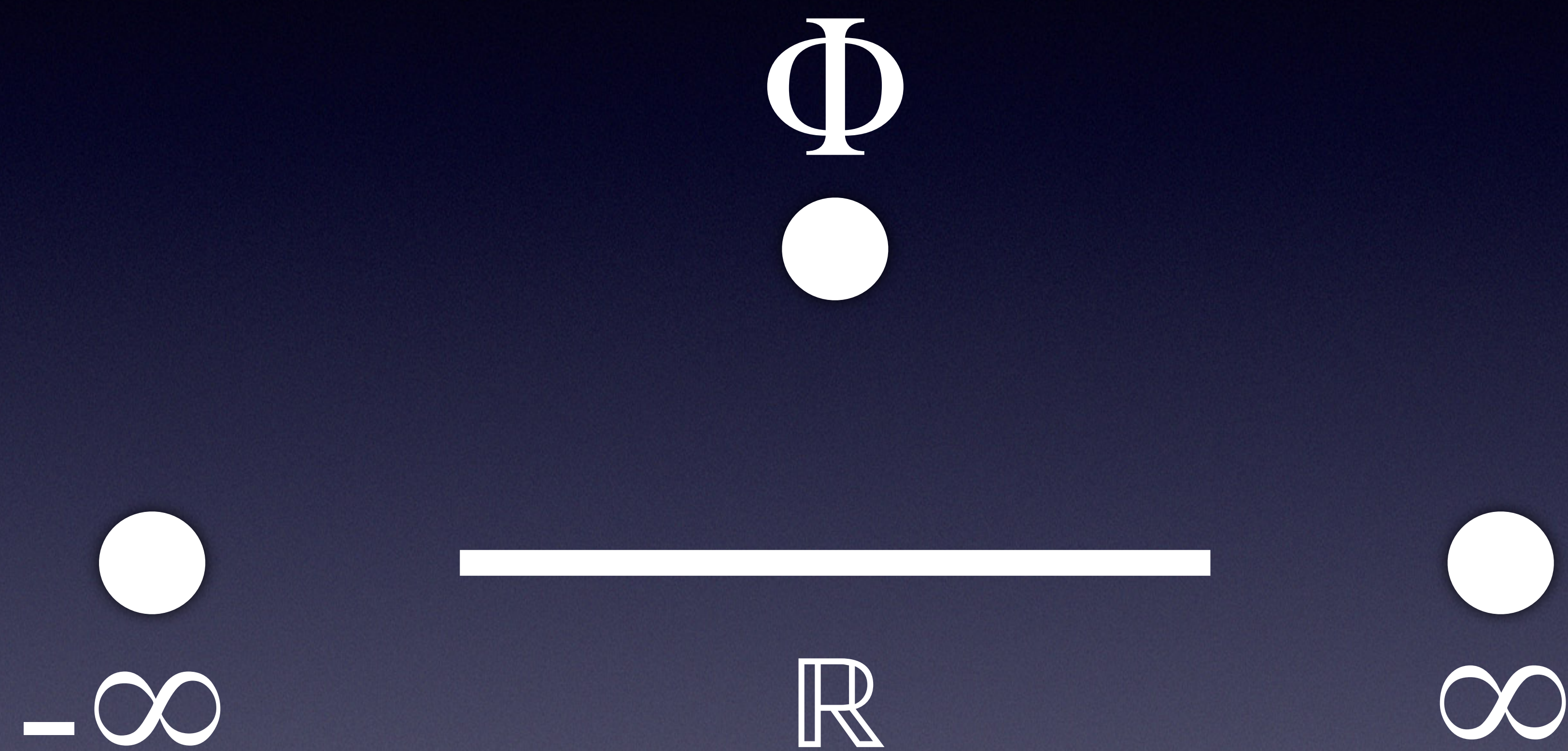
- The transreal-number line can be deduced from transreal arithmetic using epsilon neighbourhoods

Transreal-Number Line



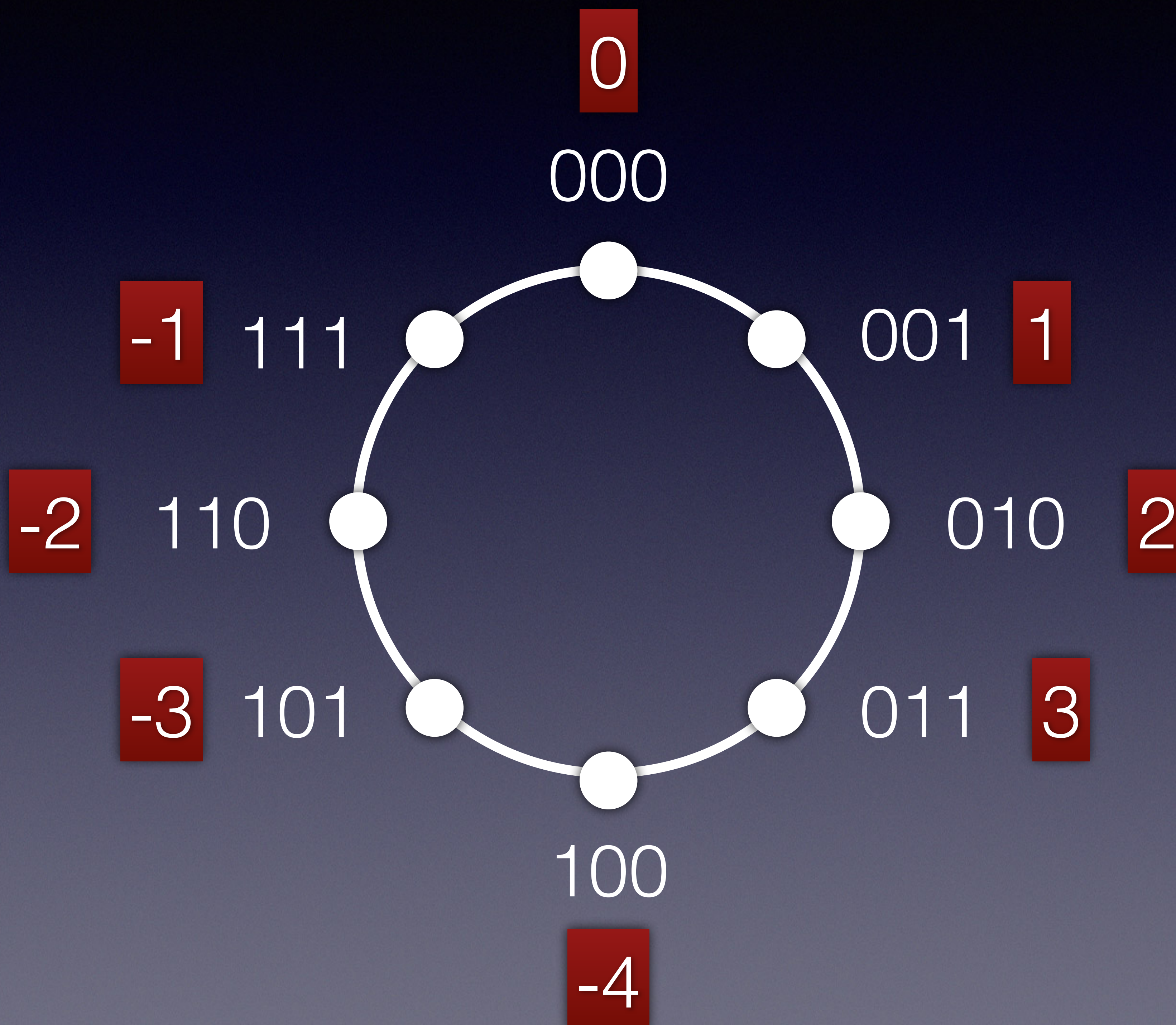
- Nullity, Φ, is unordered and lies off the transreal-number line

Transphysics



- A nullity force, Φ, has no component in the extended-real universe, $[-\infty, \infty]$, so behaves as a zero force in this universe

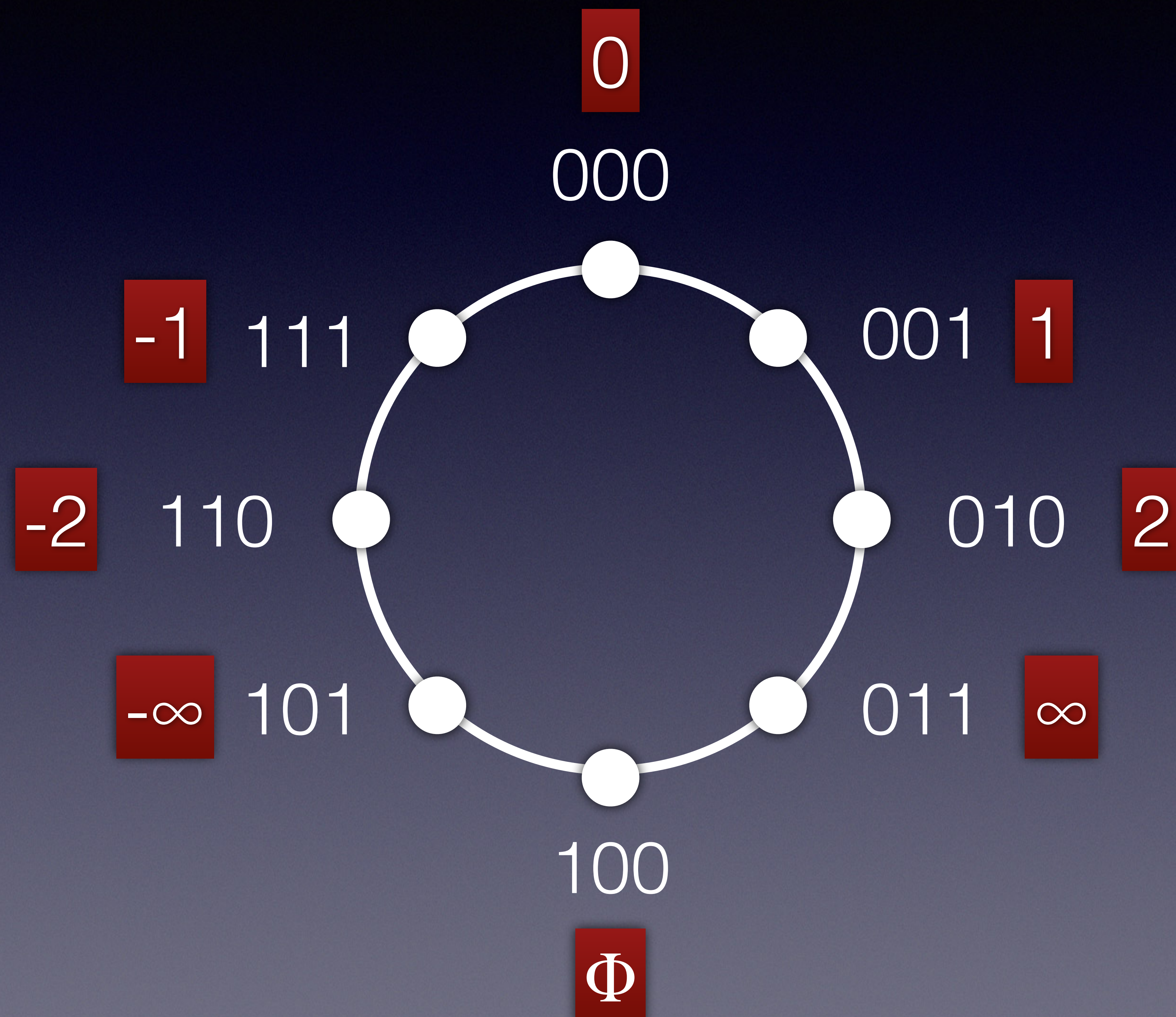
Two's Complement



Two's Complement

- Wrap-around error
- Weird-number error
- One more negative than positive numbers
- Prioritises range over correctness!
- Not embedded in the real-number or floating-point-number lines - so harder to convert real analysis into two's complement algorithms

Trans-Two's Complement



Trans-Two's Complement

- No wrap-around error
- No weird-number error
- Equal number of positive and negative numbers
- Obtains maximal range but with round-off to infinities
- Embedded in the transreal-number line so easier to convert transreal analysis into trans-two's complement algorithms

Floating-Point (NaN)

X 11111111111 XXXXXXXXXXXXXXXXXXXX

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

64-Bit Floating-Point

- 9,007,199,254,740,990 error (NaN) states
- 25,000 error (NaN) states for each star in our Milky Way galaxy!
- Not embedded in the real-number line so harder to convert real analysis into floating-point algorithms

Transfloating-Point

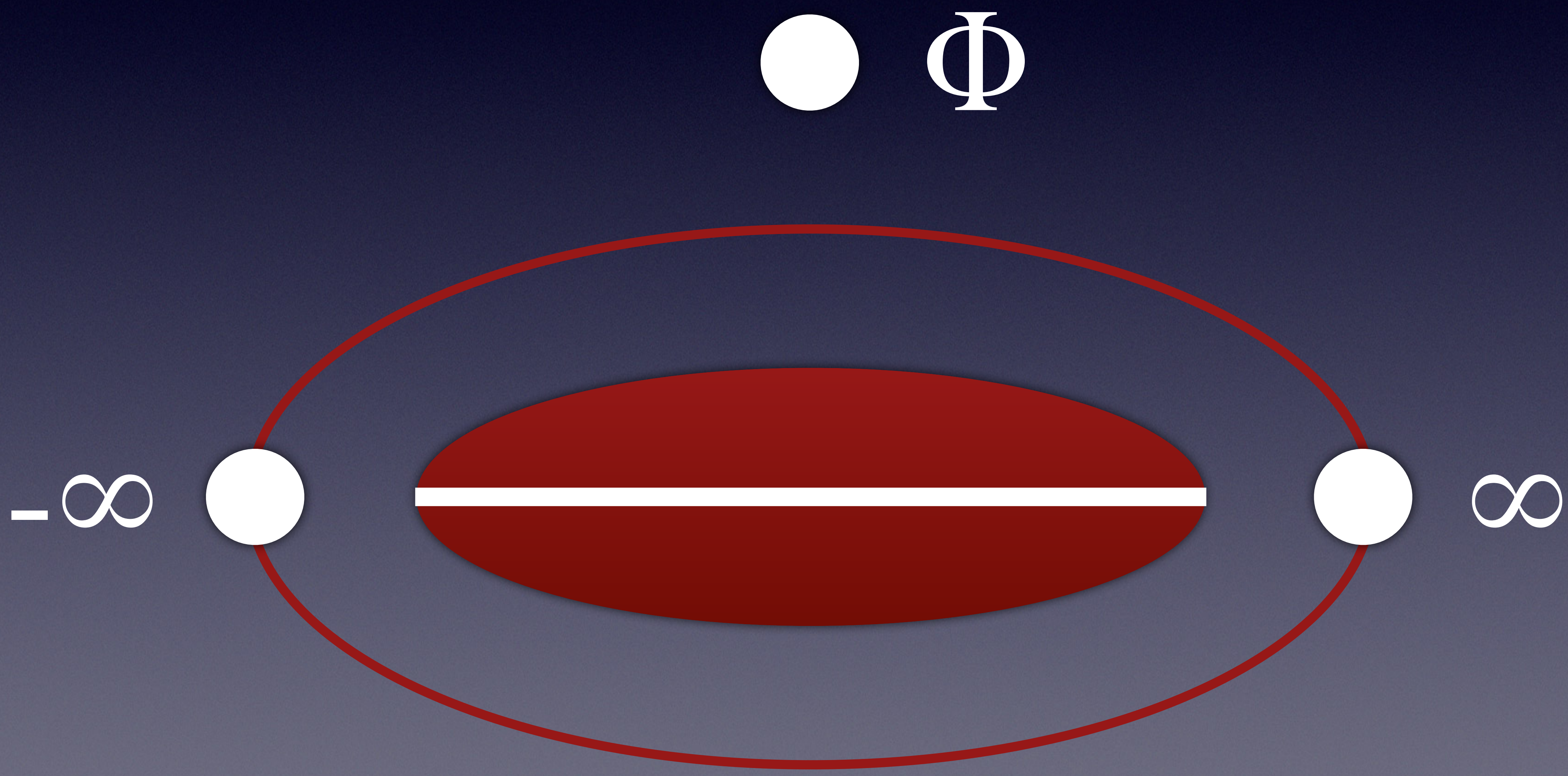
- Replace negative zero, $-0 \not\equiv 0$, with nullity Φ
- Replace NaNs by mapping $\pm\infty$ onto the extremal bit patterns
- Increment the exponent bias by one

Transfloating-Point

- Twice the range of real numbers mapped to a doubling of accuracy, by halving the smallest, representable number
- Embedded in transreal-number line so easier to convert transreal analysis into transfloating-point algorithms

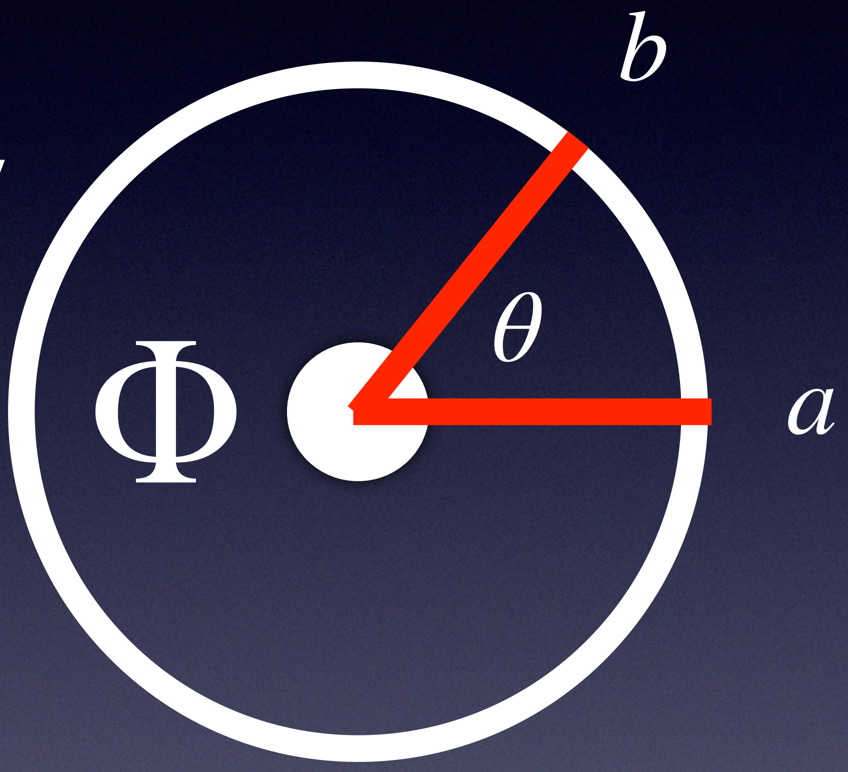
Transcomplex Plane

Revolution of the transreal number line



Transangle

Real and nullity angles are arc length divided by radius in a unit wheel

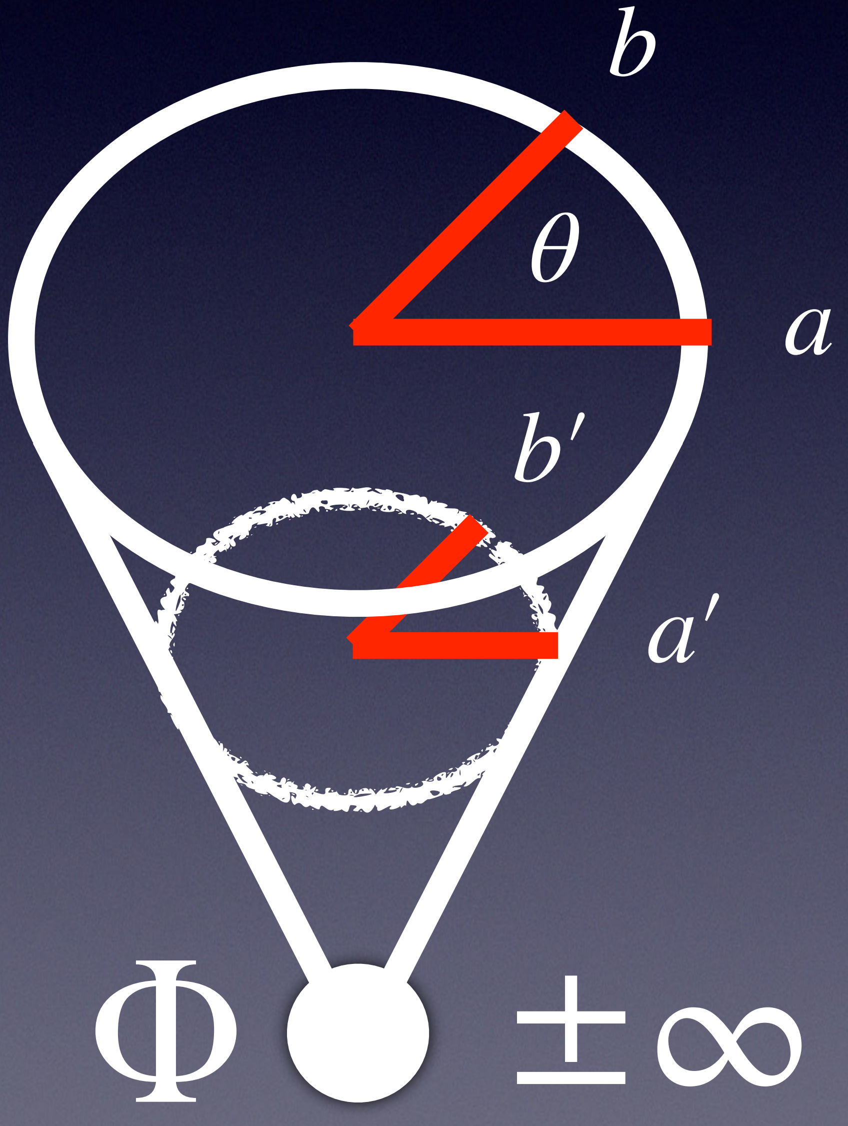


$$\Phi = \frac{0}{0} \text{ radians}$$

But where are the infinity angles?

Transangle

The infinity angles are the windings of the outer line segments at the apex of the unit cone

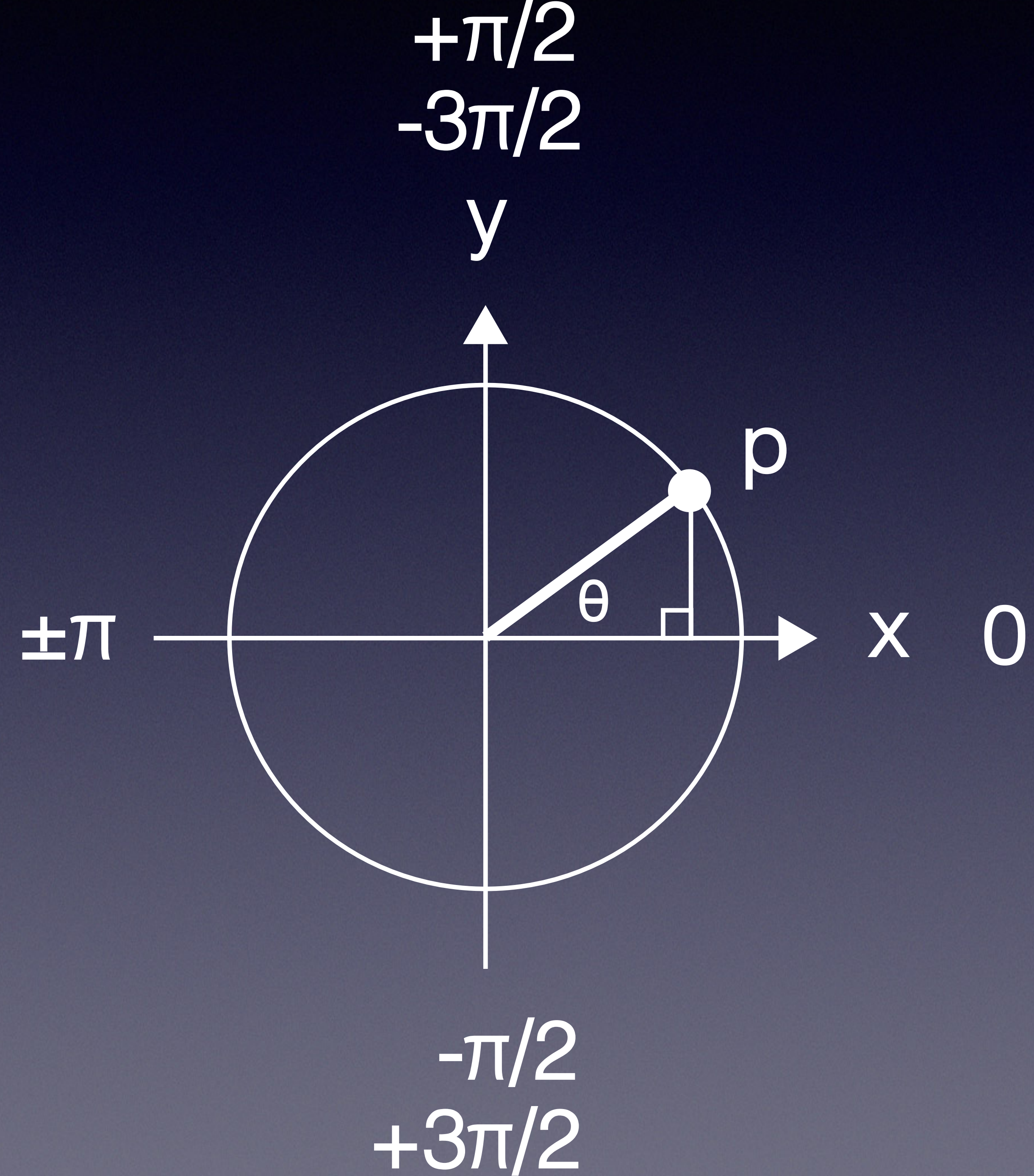


$2\infty\pi + \theta = \infty$ radians

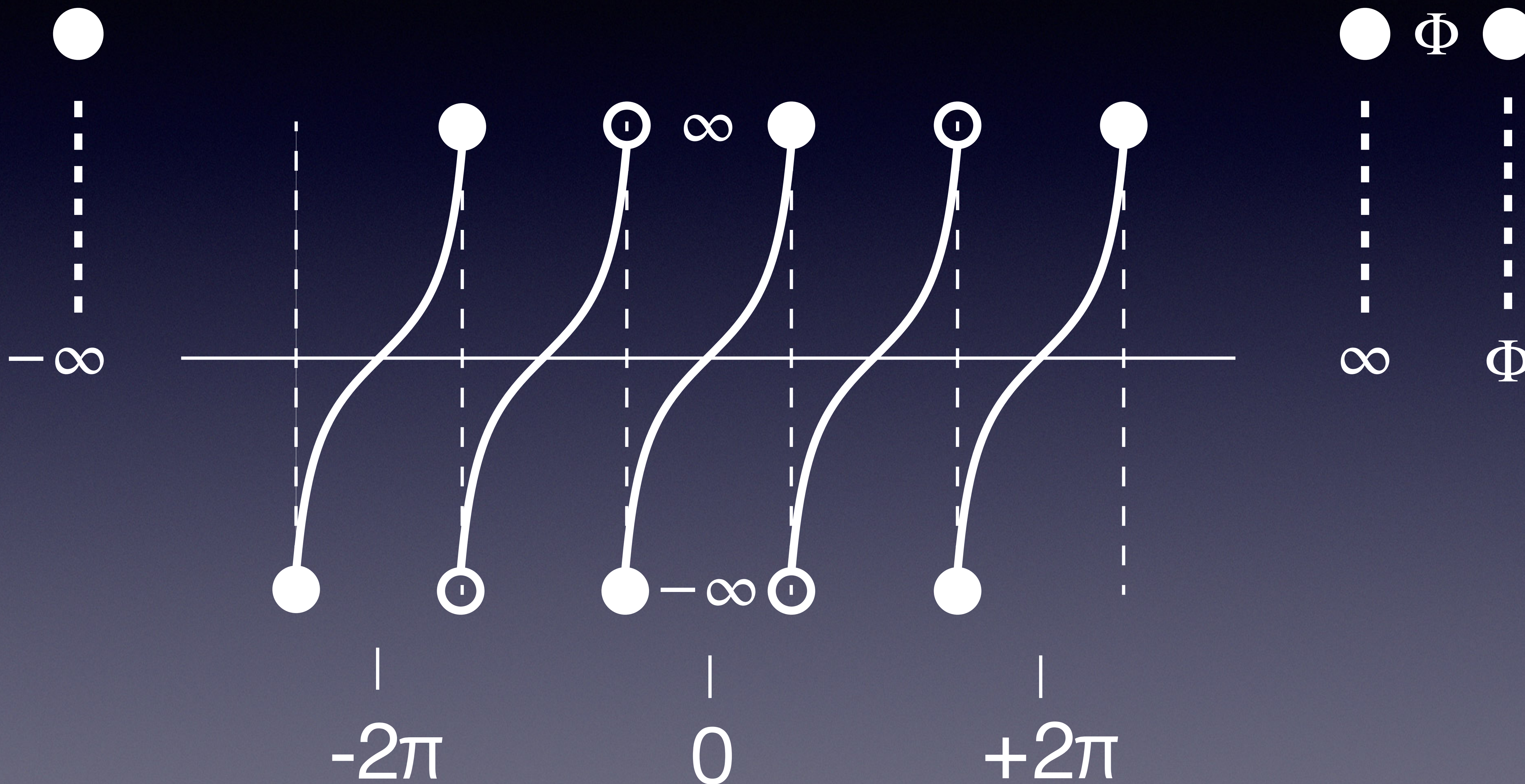
$-2\infty\pi + \theta = -\infty$ radians

Transtangent

Ancient construction of trigonometric functions as ratios, with modern annotations



Transtangent



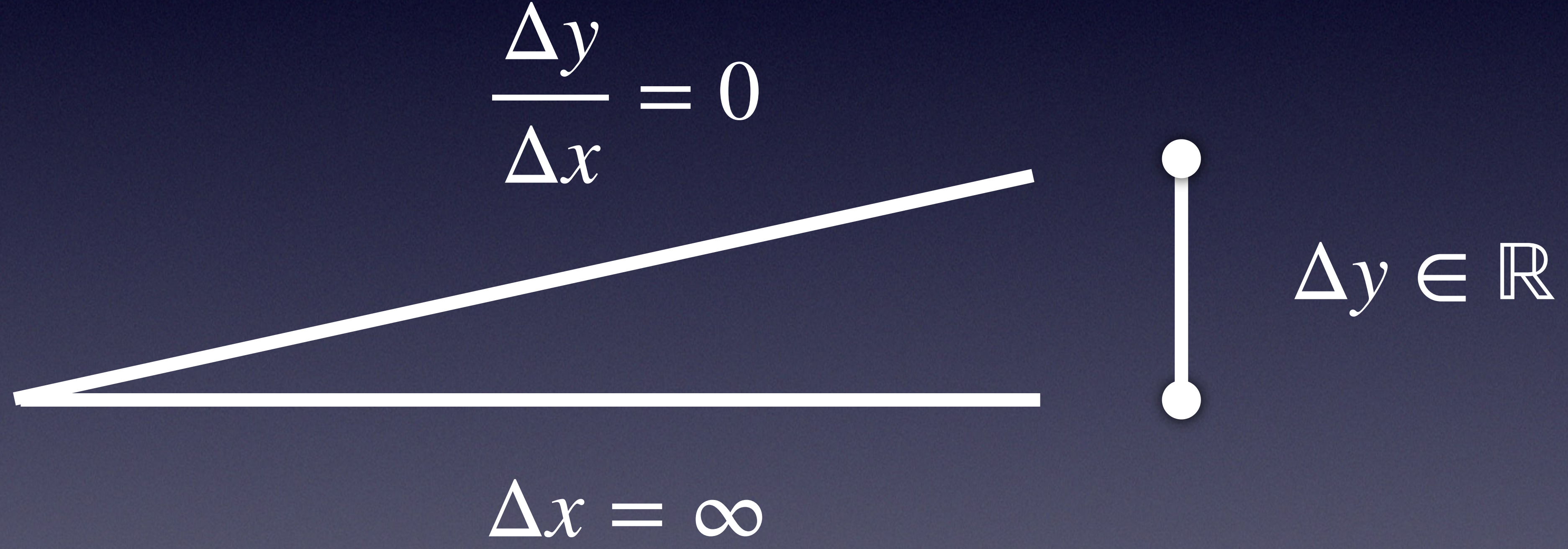
Transtangent

- Is defined for all transreal angles
- Is single valued everywhere
- The real values of the transtangent have period π
- The infinite values of the transtangent have period 2π
- The nullity values of the transtangent occur at $-\infty$, ∞, Φ with Φ being the canonical position

Transtangent

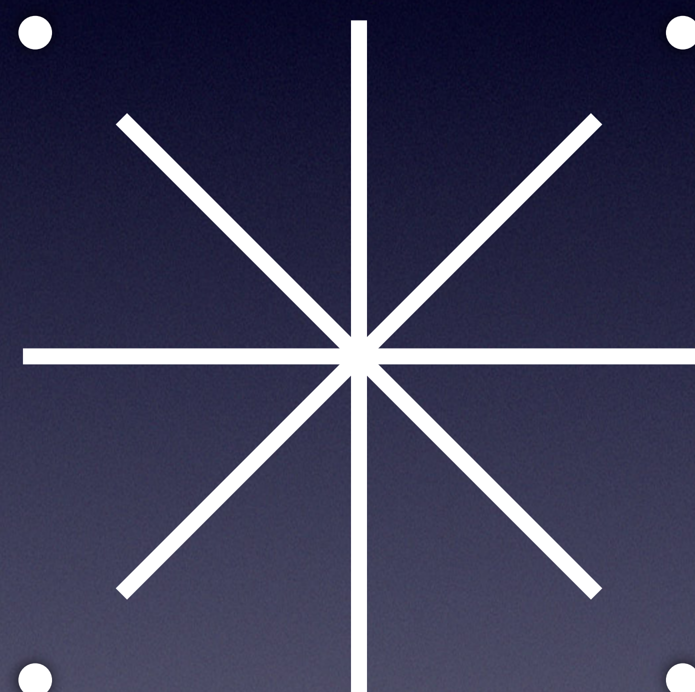
- The ancient construction of the tangent has transreal solutions that are not available to modern (non-trans) geometry and trigonometry
- The ancient construction of the tangent is identical to the transtangent
- The ancient construction of the tangent agrees with the transtangent's transpower series

Degenerate Cartesian Co-ordinates



Non-Degenerate Cartesian Co-ordinates

- $(0,\infty) = [\infty, \pi/2]$
- $(-\infty, \infty) = [\infty, 3\pi/4]$
- $(\infty, \infty) = [\infty, \pi/4]$



- $(-\infty, 0) = [\infty, \pi]$
- $(\infty, 0) = [\infty, 0]$
- $(-\infty, -\infty) = [\infty, -3\pi/4]$
- $(\infty, -\infty) = [\infty, -\pi/4]$
- $(0, -\infty) = [\infty, -\pi/2]$
- $(\Phi, \Phi) = [\Phi, \Phi]$

Transcomplex Integral

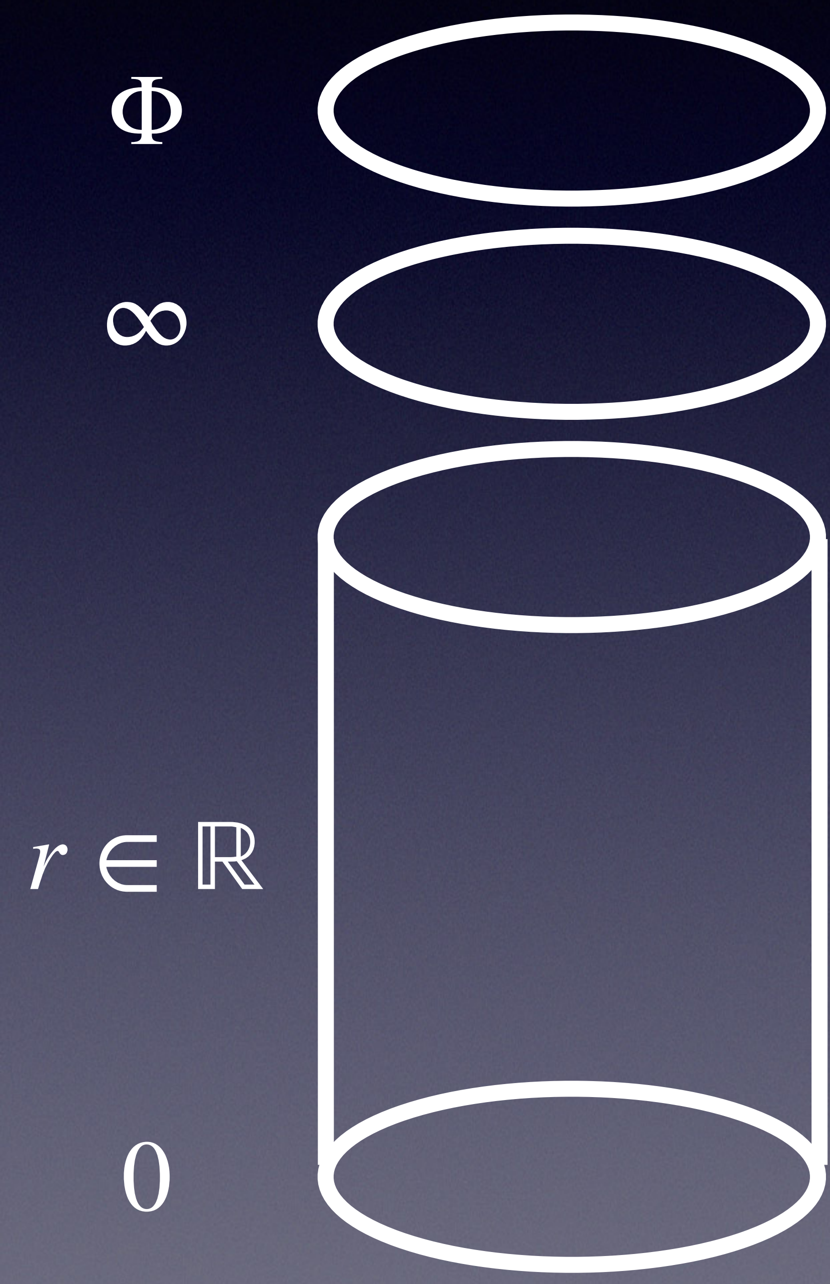
- Transcomplex numbers are represented in polar form to avoid Cartesian degeneracy
- The transomplex integral is based on the non-degenerate Cartesian co-ordinates

Transcomplex Analysis

- With the exception of the general transcomplex derivative, which has not yet been developed, all areas of complex analysis have been extended to transcomplex analysis

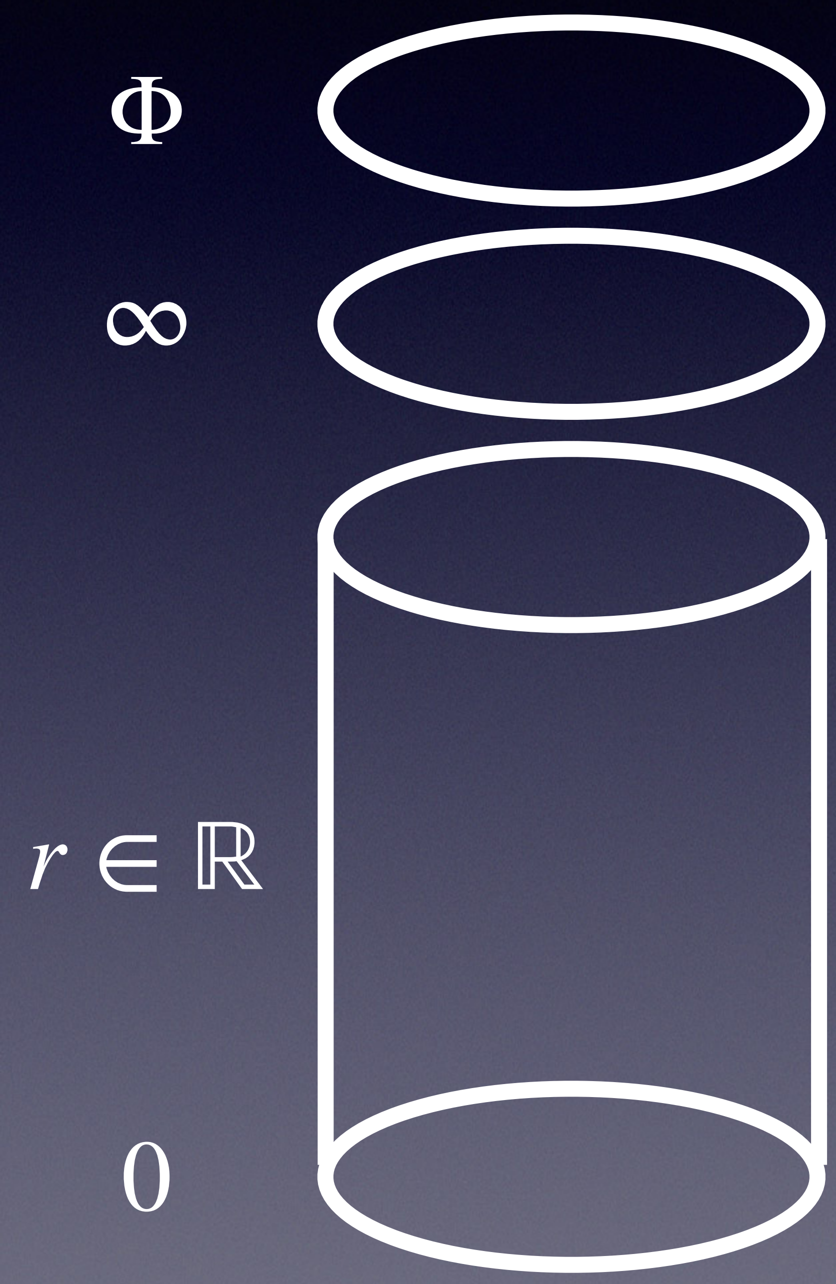
Transcomplex Arithmetic

- Solid cylinder composed of unit radius discs with zero and all positive real heights
- Unit radius disc at infinite height
- Convention - unit radius disc at nullity height placed above disc at infinity



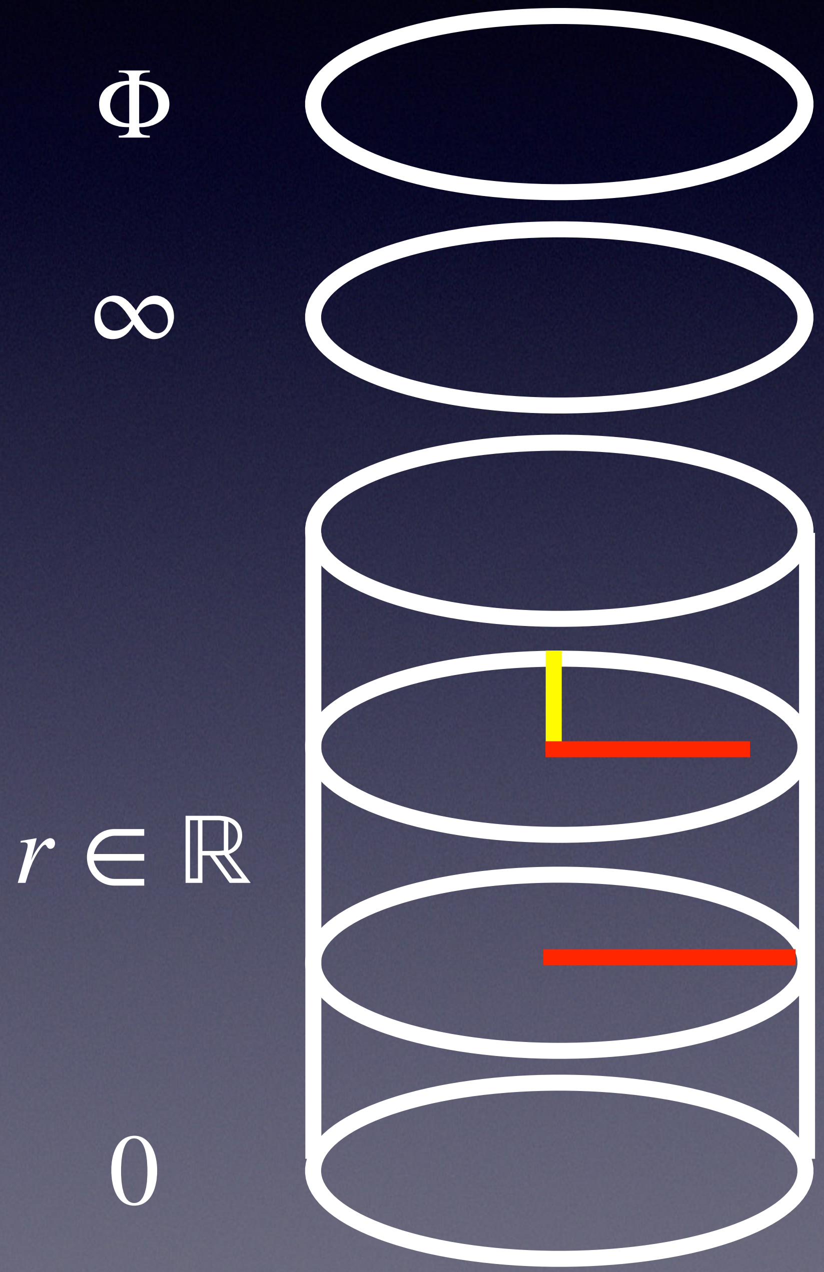
Transcomplex Multiplication and Division

- Polar multiplication and division are screws - a composition of rotation and translation



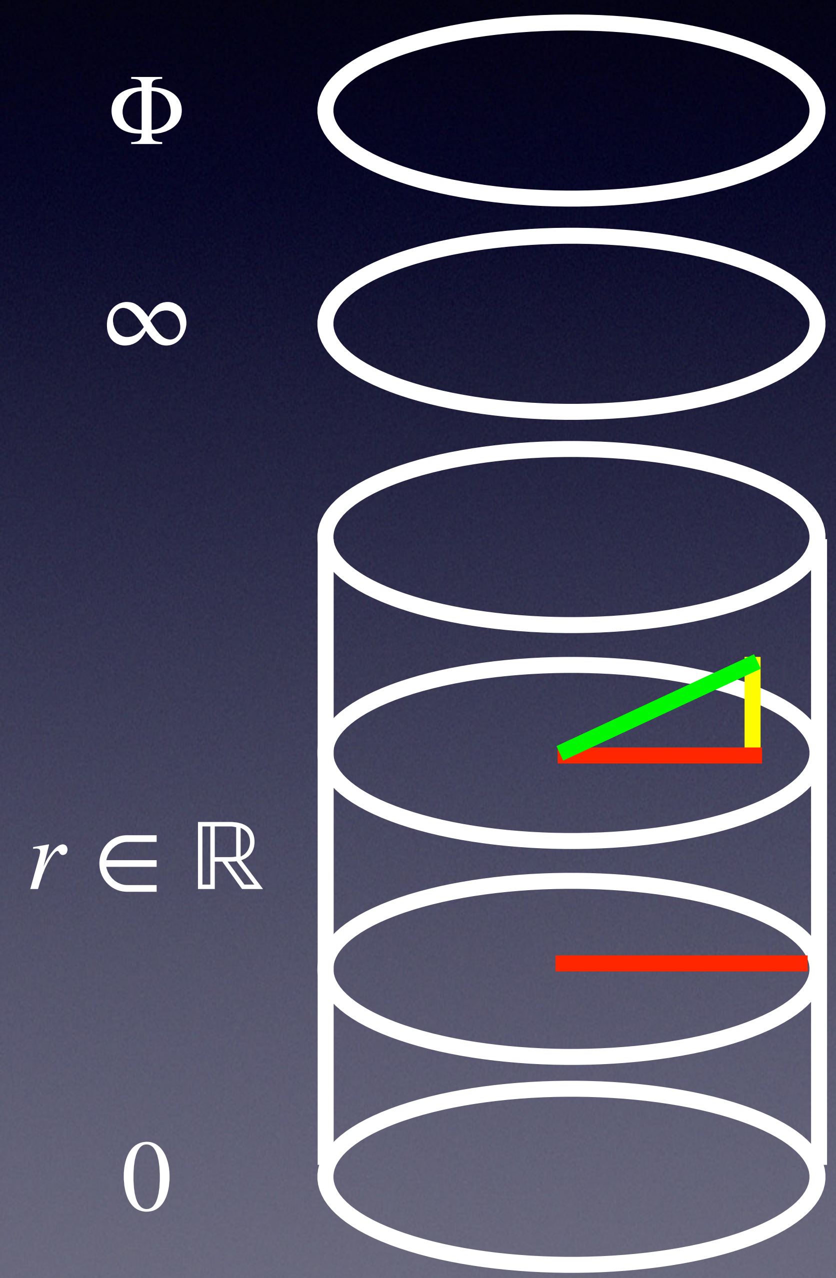
Transcomplex Addition and Subtraction

- Draw polar vector $[r, \theta]$ as a unit vector $[1, \theta]$ on the disc at height r
- Let m be the maximum height of the unit vectors in the diagram
- Scale all vectors below maximum height by $1/m$
- Move all vectors $[r/m, \theta]$ to the disc at height m



Transcomplex Addition and Subtraction

- Perform vector addition
- Scale resultant vector by m
- Redraw resultant vector $[r, \theta]$ as unit vector $[1, \theta]$ in the disc at height r
- This extends Newton's sum of directed line segments to all transreal numbers so Newton's 18th Century physics works at singularities

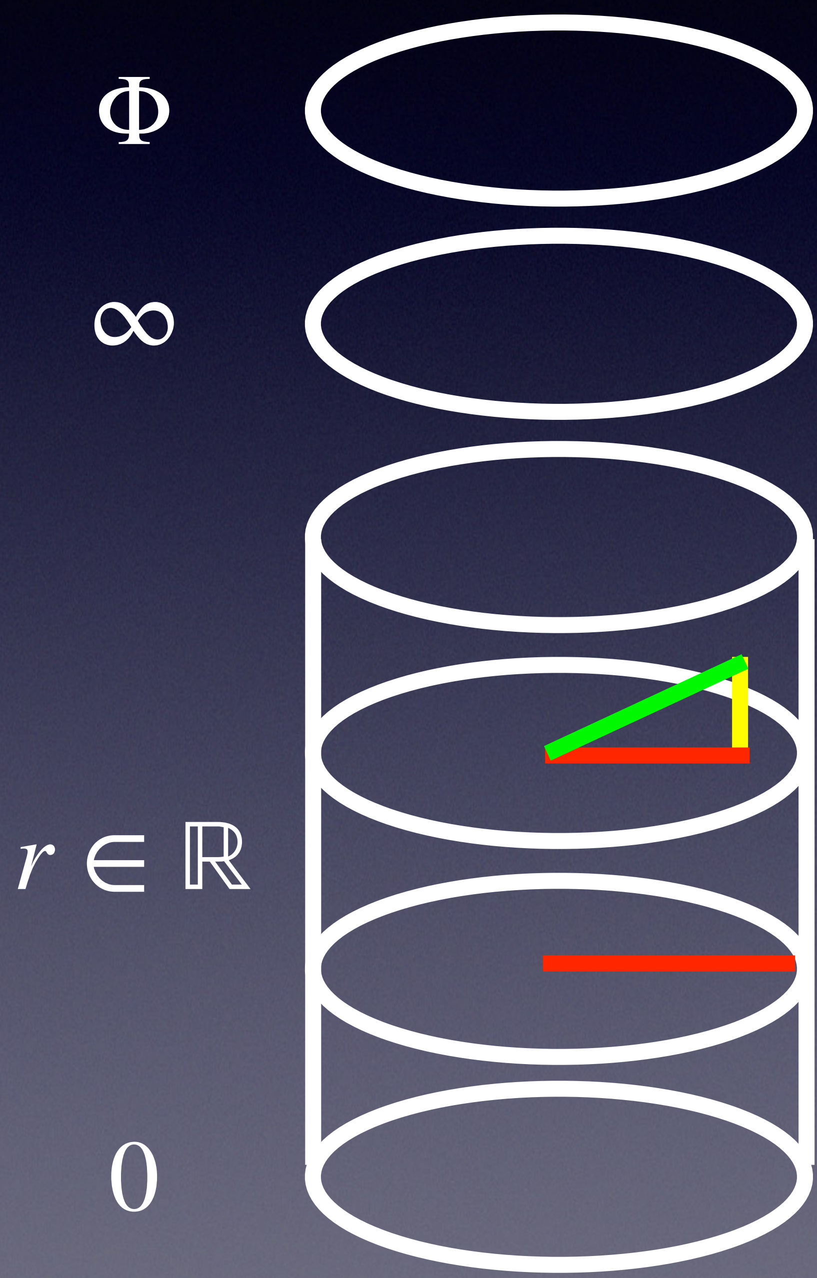


Transcomplex Addition and Subtraction

- For all transreal t
  $[\Phi, \Phi] + [t, \theta] =$
  $\Phi\{[1,\Phi] + [t/\Phi, \theta]\} =$
  $\Phi\{[\Phi, \Phi] + [\Phi, \Phi]\} =$
  $[\Phi, \Phi]$

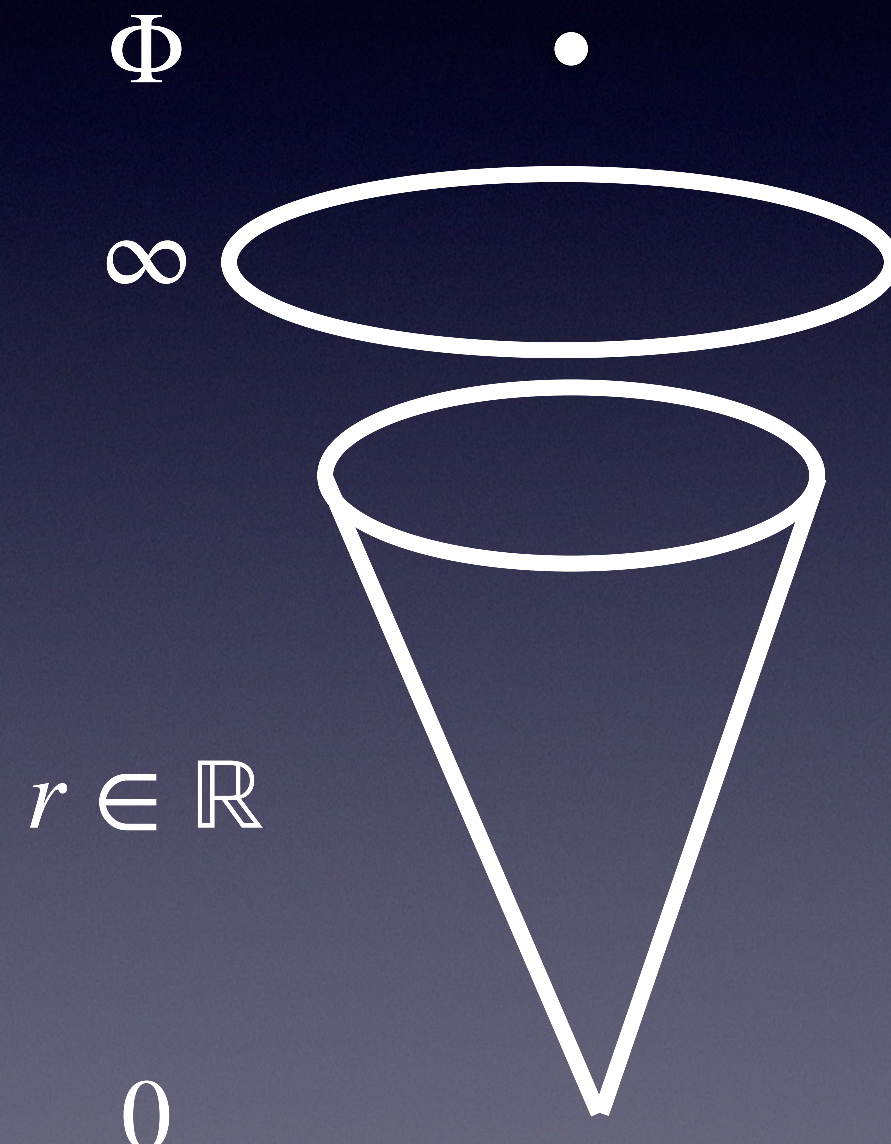
- $[\infty, \theta] - [\infty, \theta] =$
  $\infty\{[1,\theta] - [1,\theta]\} =$
  $\infty[0,0] =$
  $[\Phi, \Phi]$

- For all real r
  $[\infty, \theta_1] + [r, \theta_2] =$
  $\infty([1,\theta_1] + [r/\infty, \theta_2]) =$
  $\infty([1,\theta_1] + [0,0]) =$
  $\infty[1,\theta_1] =$
  $[\infty, \theta_1]$

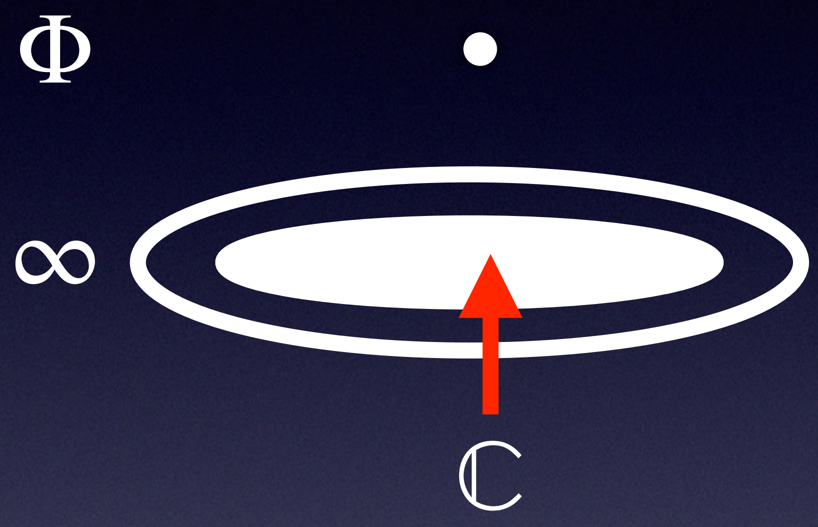


Transcomplex Cone

- Dilatate all of the discs by their height



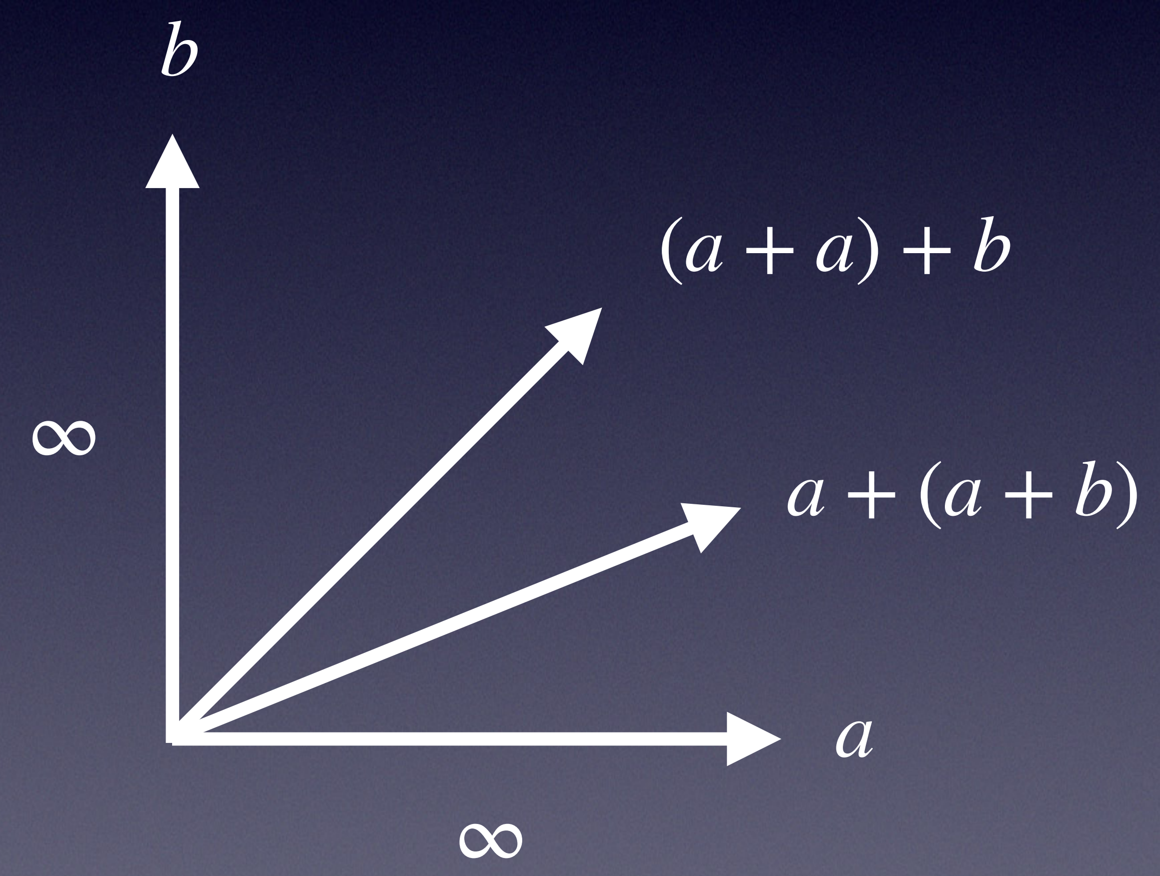
Transcomplex Plane



- Project onto a plane

Non-Associativity

- After renormalisation, to the disc at height infinity, the sum of infinite vectors can be non-associative
- Before renormalisation, the sum of arbitrarily many infinite vectors is associative

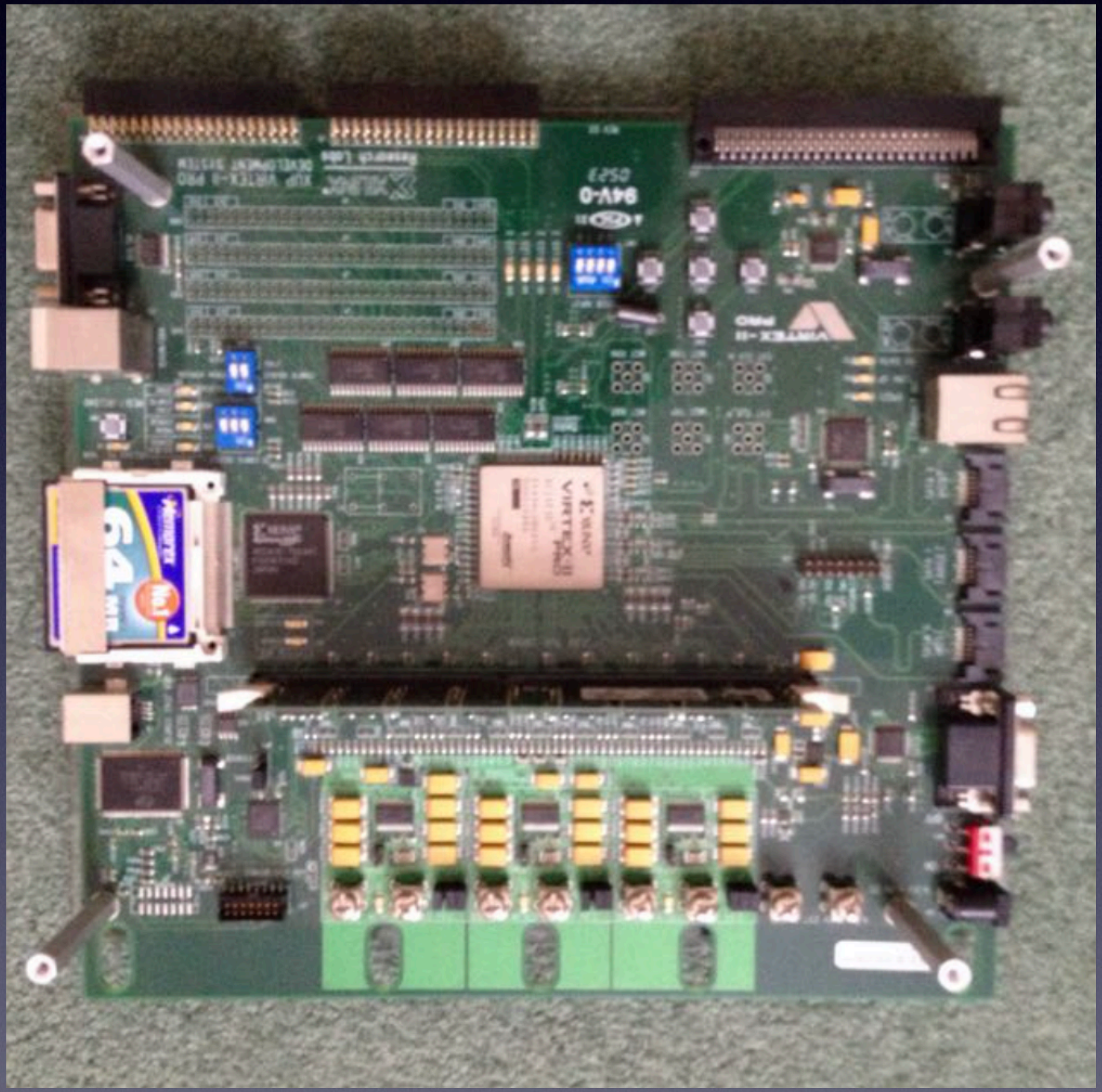


Algebraic Structure

- Multiplying by infinity can be non-distributive
- Adding infinite vectors can be non-associative
- Nonetheless transreal and transcomplex arithmetic are total

Transcomputation

- Prototype transcomputer in hardware and software emulation



Architectural Prototype

- Token = 12-bit header + 80-bit transfloat datum
- 64 k mills per chip
- 2 M mills per board
- 16 M mills per cabinet
- 20 kW per unweighted Wassenaar Peta FLOP (PWFLOP)

I/O

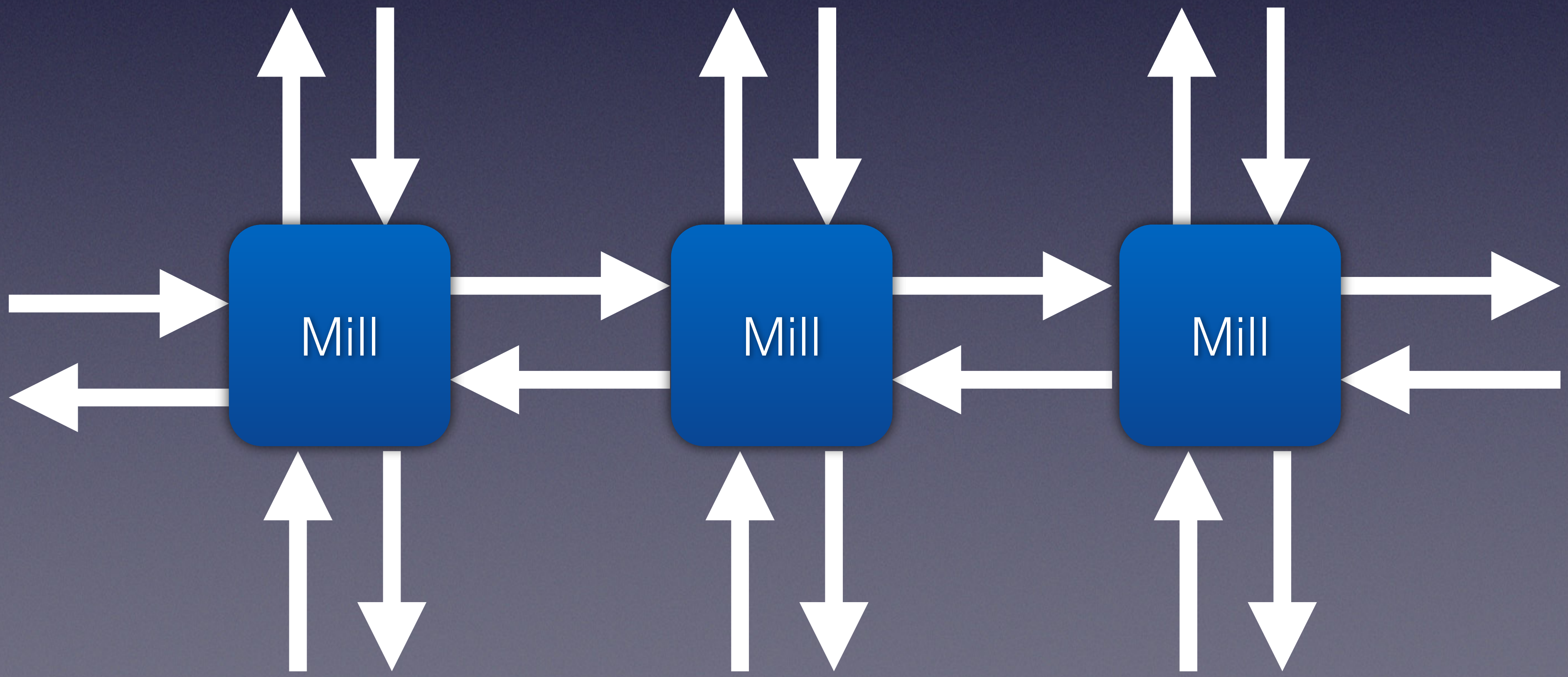
- Systolic arrays have one dimensional I/O which has linear scaling and is impossible to fabricate
- Architectural prototype uses zero dimensional I/O which has constant scaling and can be fabricated

Relative Addressing

- Fixed size, relative address implements an address horizon in an arbitrarily large machine and maintains constant computational efficiency regardless of the size of the machine
- Small horizon keeps the token header small

Processor Grid

- Square grid of mills
- Pipelined communication not just nearest neighbour



Slipstream

- A grid of mills may be arranged in any dimensionality of space (2D is convenient for chips!)

- The nodes of the grid are coloured by the configuration state of the mills

- A Turing program is a directed graph in a grid

- A slipstream program is an acyclic graph in a grid

Slipstream

- Slipstream programs execute in a cadence (period) of the longer of the input and output times

- Programs with shared data, such as molecular dynamics, may have many copies of a program that share data so the average cadence is less than one and the limit of the cadence, with increasing machine size, can be zero!

Slipstream

- A practical slipstream machine cannot achieve a cadence of zero
- But the ratio of the execution time of a practical slipstream machine versus a practical von Neumann serial or parallel machine can be infinity - slipstream dominance
- Quantum computers can be slipstreamed

FFT

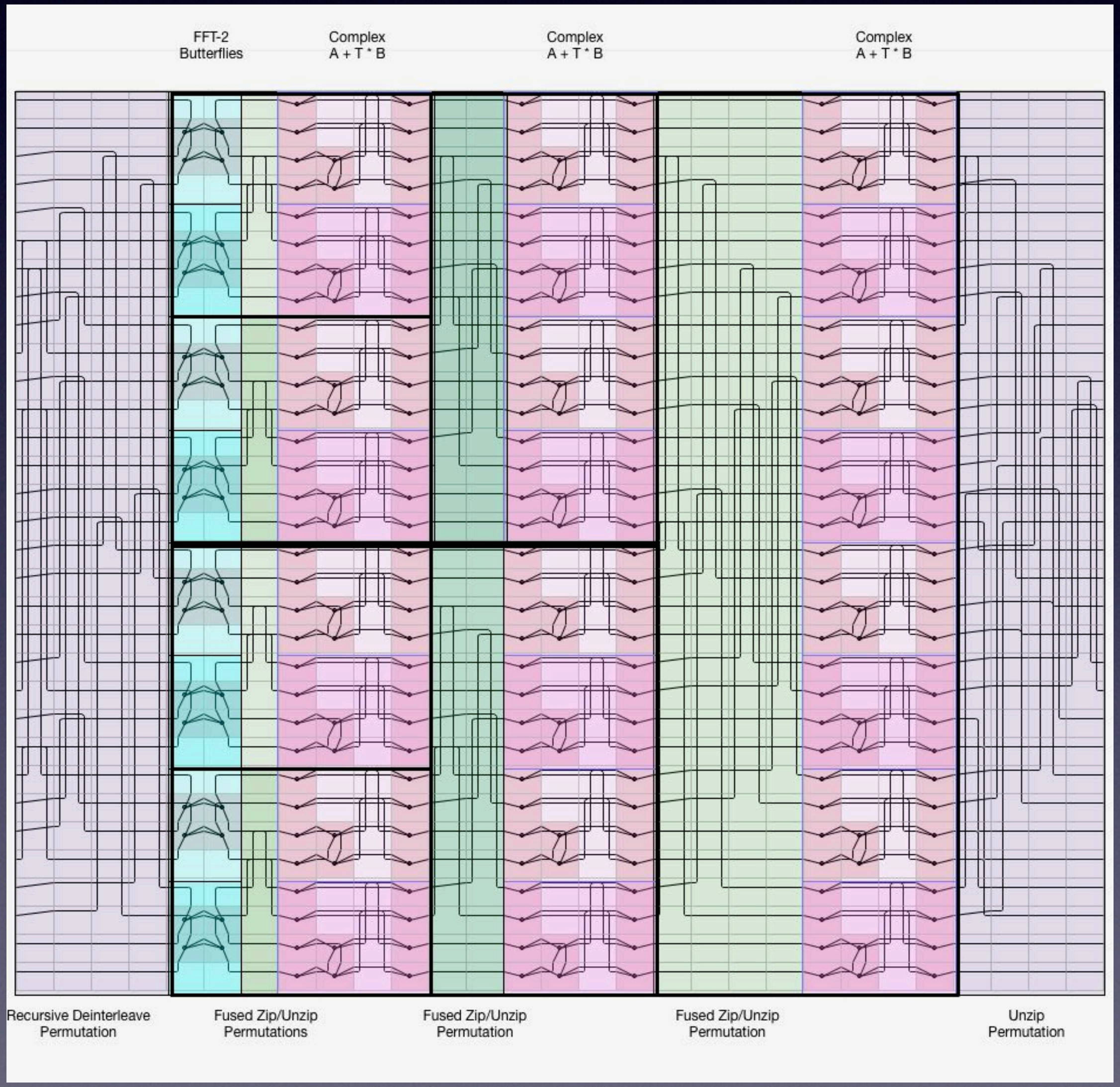
- Fourier Transform (FT) time order $O(n^2)$
- Fast Fourier Transform (FFT) time order $O(n \log n)$
- Slipstream FFT time order $O(n)$

Slipstream FFT

- Multitap liner-time processing:
- Sharpened radar image
- Detected objects
- Identified objects

Slipstream FFT

- Turing-complete compiler
- Optimisers
- Optimise cadence



Summary

- If Newton's arithmetica universalis is restricted to real arithmetic then his 18th Century physics fail at singularities

- If Newton's arithmetica universalis is stated in transreal arithmetic then his 18th Century physics succeed at singularities

Summary

- Totality - every function is a total function
- It is not necessary to waste states - so hardware and software can be more efficient
- Market opportunity to sell transcomputing FPGAs, ASIC IP cores, turnkey systems
- Marketing advantage of selling computers without an astronomical number of errors

Summary

- Transfloating-point arithmetic is up to twice as accurate as floating-point arithmetic, using the same number of bits
- Every syntactically correct program is semantically correct - except for physical intervention, programs cannot crash
- On average pipelined programs, with shared data, can complete execution in less than one clock cycle

Summary

- There are many areas of transmathematics, its application and meta theory that are not presented here

Summary

- More than 10 people have published transmathematics in conference proceedings and journals, about half are computer scientists
- Transmathematica journal
- Transmathematica conference
- Transmathematica society holds weekly Skype meetings on Mondays at 17.00 London time